A PROOF FOR THE QUASISTEADY METHOD OF SOLVING THE STEFAN PROBLEM

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The location limit for the interphase boundary is found in the region exterior to a sphere with a finite radius. It is shown that the solution to the Stefan problem for this region by the method of quasisteady states approaches the same limit as $t \to \infty$.

I. Let region G bounded by two concentric spheres with the respective radii R_0 and R_1 contain a solid substance at temperature T_0. At the instant of time t = 0 heat is supplied to the inner sphere so that a constant temperature $T_G > T_m$ is maintained on it. At the same time, heat is removed from the outer sphere so that the initial temperature $T_0 < T_m$ is maintained here. Obviously, the solid medium inside region G will begin to melt and the interphase boundary will be a sphere with the radius $\xi = \xi(t, R_1)$. Then

$$\eta(t) = \lim_{R_1 \to \infty} \xi(t, R_1) \tag{1}$$

will be the coordinate of the interphase boundary within the region outside the sphere with radius R_0. Reaching the limit R_1 is possible, because at a finite time function $\eta(t)$ is bounded and $\xi(t, R_1^{(1)}) < \xi(t, R_1^{(2)})$ if $R_1^{(1)} < R_1^{(2)}$. We will prove that $\eta(t)$ is bounded at $t \to \infty$. For this, we must show that there exists a limit

$$\lim_{t \to \infty} \eta(t) = \lim_{t \to \infty} \lim_{R_1 \to \infty} \xi(t, R_1) = \overline{R}. \tag{2}$$

In order to prove this, we consider the expression

$$\overline{\overline{R}} = \lim_{R_1 \to \infty} \lim_{t \to \infty} \xi(t, R_1). \tag{3}$$

Let $\bar{\xi}(R_1) = \lim_{t \to \infty} \xi(t, R_1)$ represent the steady-state solution to the Stefan problem for region G. It can be easily obtained by stipulating the steady-state temperature distribution in zones with different phase contents and by using the condition of equal thermal fluxes at the boundary when $R = \xi$ [1]. This yields an expression for the steady-state boundary ξ:

$$\bar{\xi}(R_1) = \lim_{t \to \infty} \xi(t, R_1) = \frac{R_0 R_1 [\lambda_1 (T_G - T_m) + \lambda_2 (T_m - T_0)]}{R_0 \lambda_1 (T_G - T_m) + R_1 \lambda_2 (T_m - T_0)}. \tag{4}$$

With (4), we have

$$\overline{\overline{R}} = \lim_{R_1 \to \infty} \lim_{t \to \infty} \xi(t, R_1) = R_0 \left[1 + \frac{\lambda_1}{\lambda_2} \frac{(T_G - T_m)}{(T_m - T_0)} \right]. \tag{5}$$

We will now show that the limits in (2) and (3) can be switched around, i.e., that R exists and $\overline{R} = \overline{\overline{R}}$. For this we must show [2] that $\xi(t, R_1)$ converges uniformly at least with respect to one variable. We consider the family of functions $\{\xi_n(t)\}$:

$$\xi_n(t) = \xi[t, R_0(1+n)];\ n = 1, 2, \ldots \tag{6}$$

Functions $\xi_n(t)$ have the following properties:

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1) $\xi_n(t)$ increases monotonically;

2) at any fixed time t*

$$\xi_n(t^*) < \xi_{n+1}(t^*); \tag{7}$$

3) $d\xi_n(t)/dt$ decreases monotonically;

4) at any fixed time t* > 0

$$\frac{d\xi_n(t^*)}{dt} < N, \tag{8}$$

where N is a number of independent of n.

Properties 1)-4) follow from the most general energy concepts. From the first two, in turn, and from (5) follows the uniform boundedness of the family of functions $\{\xi_n(t)\}$, while from the last two follows the equidegree continuity of the family of functions $\{\xi_n(t)\}$ on any interval $[t^* > 0, \infty)$. Consequently, the family of functions $\{\xi_n(t)\}$ satisfies Arzela's theorem [4]. Arzela's theorem and (7) prove the uniform convergence of the family of functions $\{\xi_n(t)\}$ on any interval $[t^* > 0, \infty)$.

Consequently,

$$\overline{R} = \bar{\bar{R}} = R_0\left[1 + \frac{\lambda_1}{\lambda_2}\frac{(T_G - T_m)}{(T_m - T_0)}\right]. \tag{9}$$

If $\lambda_1 = \lambda_2$, then

$$\overline{R} = R_0\left(1 + \frac{T_G - T_m}{T_m - T_0}\right). \tag{10}$$

One can also arrive at (10) considering the limit location of the T_m isotherm isotherm in the problem without phase transition.

Thus, in the Stefan problem with center symmetry, within the region exterior to a sphere with the radius R_0 there is a limit for the location of the interphase boundary as $t \to \infty$ and this limit is equal to expression (9).

II. We will now consider the solution to the Stefan problem for the region exterior to a sphere with the radius R_0 and will use here the method of quasisteady states [1]. The initial and the boundary conditions are the same as in ¶ I. According to [1], the temperatures in the zones with different phase contents are specified in the form:

$$\left.\begin{aligned} T_1(t, r) &= T_G + \frac{T_G - T_m}{\eta - R_0}\left(\frac{R_0}{r} - 1\right)\eta; \; R_0 \leqslant r \leqslant \eta, \\ T_2(t, r) &= T_0 + \frac{(T_m - T_0)}{r}\eta \operatorname{erfc}\left[\frac{r-\eta}{2\sqrt{\varkappa_2 t}}\right]; \; r \geqslant \eta. \end{aligned}\right| \tag{11}$$

The differential equation describing the movement of the interphase boundary will be obtained from the Stefan condition at the boundary:

$$\lambda_2 \left.\frac{\partial T_2}{\partial r}\right|_{r=\eta} - \lambda_1 \left.\frac{\partial T_1}{\partial r}\right|_{r=\eta} = L\rho_2 \frac{d\eta}{dt}. \tag{12}$$

Inserting (11) into (12) yields

$$\frac{d\eta}{dt} = \frac{A}{\eta(\eta - R_0)} - \frac{B}{\eta} - \frac{C}{\sqrt{t}}; \; \eta(0) = R_0, \tag{13}$$

where

$$A = \frac{\lambda_1(T_G - T_m)R_0}{L\rho_2}; \; B = \frac{\lambda_2(T_m - T_0)}{L\rho_2}; \; C = \frac{\lambda_2(T_m - T_0)}{L\rho_2\sqrt{\pi\varkappa_2}}. \tag{14}$$

We rewrite (13) as

$$\frac{d\eta}{dt} = \frac{A\sqrt{t} - B(\eta - R_0)\sqrt{t} - C\eta(\eta - R_0)}{\eta(\eta - R_0)\sqrt{t}}; \quad \eta(0) = R_0. \tag{15}$$

It can be shown that the solution $\eta(t)$ to this nonlinear differential equation, with the given initial condition, will be a monotonically increasing function. It appears from (15), however, that, in order to satisfy the condition $d\eta/dt > 0$ at any instant of time, function $\eta(t)$ must be upper bounded, i.e., $\eta(t)$ must have a limit as $t \to \infty$. This limit will be found by letting the time approach infinity on the right-hand side of Eq. (15) and by making use of (14):

$$\bar{R} = \lim_{t \to \infty} \eta(t) = R_0 + \frac{A}{B} = R_0 \left[1 + \frac{\lambda_1}{\lambda_2} \frac{(T_G - T_m)}{(T_m - T_0)} \right]. \tag{16}$$

Expression (16) obtained by the method of quasisteady states is the same as expression (9) obtained in the exact formulation. Consequently, the method of quasisteady states is an asymptotically stable one.

We note, in conclusion, that our results remains valid also for a boundary condition of the third kind and the limit location of the interphase boundary will be the same as for a boundary condition of the first kind. Furthermore, these results can be extended to the case where the temperature at the boundary is a function of time with a bounded asymptote $\lim_{t \to \infty} T_G(t) = T_G > T_m$.

NOTATION

t is the time;
r is the radial coordinate;
T is the temperature;
T_m is the melting point;
λ is the thermal conductivity;
$\varkappa$ is the thermal diffusivity;
ρ is the density;
c is the specific heat;
L is the latent heat of melting.

Subscripts

1 denotes the liquid phase;
2 denotes the solid phase.

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